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A convergence analysis of a numerical method for solving the balance equation.

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Summary.

In this report the convergence of an iterative method for solving the nonlinear balance equation is analyzed. It is shown that this iterative method will be convergent if a sufficiently accurate initial approximation is used and if the subsequent iterates satisfy the ellipticity condition. Otherwise if the successive approximate solutions do not all satisfy the ellipticity condition the method may be divergent. We shall describe what can be done to make the method convergent in this case. Experimental results are given.

1. Introduction.

The balance equation obtained by applying the two-dimensional divergence operator to the primitive equations of horizontal motion and retaining only the divergence-free part of the velocity in the resulting equation may be written as

$$f\,\nabla^2\psi + 2\,(\psi_{xx}\,\psi_{yy} - \psi_{xy}^2) + \nabla f.\nabla\psi - \nabla^2\Phi = 0, \qquad (1.1)$$

where ψ is the stream function, f the Coriolis parameter, Φ the geopotential and $\nabla^2\psi$ the vorticity. The condition that equation (1.1) be elliptic is (see Árnason, 1958, p. 221)

$$(f + 2\psi_{xx})\,(f + 2\psi_{yy}) - 4\,\psi_{xy}^2 > 0. \qquad (1.2)$$

In combination with equation (1.1) condition (1.2) can be written in the form

$$\nabla^2\Phi + \frac{f^2}{2} - \nabla f.\nabla\psi > 0\,. \qquad (1.3)$$

Condition (1.2) implies that there are two distinct solutions of (1.1) for given boundary values of ψ, one for which the quantities $f + 2\psi_{xx}$ and $f + 2\psi_{yy}$ are both positive and the other for which these quantities are negative. Since it is generally observed that the absolute vorticity

$$f + \nabla^2\psi$$

is positive (Árnason, 1958, p. 221) we impose apart from (1.2) or (1.3) the conditions

$$f + 2\psi_{xx} > 0 \quad \text{and} \quad f + 2\psi_{yy} > 0. \qquad (1.4)$$

Using the identity

$$4\psi_{xx}\,\psi_{yy} = (\nabla^2\psi)^2 - (\psi_{xx} - \psi_{yy})^2\,,$$

equation (1.1) reads

$$\tfrac{1}{2}\left[(\nabla^2\psi)^2 - (\psi_{xx} - \psi_{yy})^2 - 4\psi_{xy}^2\right] + f\,\nabla^2\psi + \nabla f.\nabla\psi - \nabla^2\Phi = 0 \qquad (1.5)$$

or with (1.4)

$$\nabla^2\psi = -f + \sqrt{2\,\nabla^2\Phi + f^2 + A^2 + B^2 - 2\,\nabla f.\nabla\psi}\ , \qquad (1.6)$$

where

$$A = \psi_{xx} - \psi_{yy} \quad , \quad B = -2\,\psi_{xy}\ .$$

In the following we shall analyze the convergence behaviour of a commenly used iterative method (see for instance Shuman (1957a) or Miyakoda (1956)) for solving equation (1.6). 'Arnason (1958) proposed an iterative method to solve equation (1.5). He also showed that this method might be divergent if the relative vorticity takes large positive values. Paegle and Tomlinson (1975) introduced a modification of this method which is convergent in case of large positive vorticity. For experiments the reader is referred to page 534 of their article. Applying the same modification to the method of Shuman (1957a) the modified method also converged where the original method diverged. But no such vorticity criterion was appearant. In this report we shall try to find out what might be the reason of the convergence behaviour of the Shuman and modified Shuman method.

2. The method of solution.

We consider a sequence of functions $\psi^{(i)}$ $(i=0,1,2,\ldots)$, determined by

$$\nabla^2\psi^{(n)} = -f + \sqrt{2\nabla^2\Phi + f^2 + A^2_{n-1} + B^2_{n-1} - 2\nabla f.\nabla\psi^{(n-1)}} \qquad (2.1)$$

with $A_{n-1} = \psi^{(n-1)}_{xx} - \psi^{(n-1)}_{yy}$, $B_{n-1} = -2\psi^{(n-1)}_{xy}$, which satisfy the same boundary conditions as ψ from equation (1.1). If the difference $\varepsilon^{(n)} = \psi^{(n)} - \psi^{(n-1)}$ converges towards zero, the sequence of approximate solutions from (2.1) will converge towards the solution ψ of equation (1.1). From (2.1) we have

$$\nabla^2\varepsilon^{(n+1)} = F(\psi^{(n)}) - F(\psi^{(n-1)}),$$

where

$$F(\psi^{(n)}) = \sqrt{2\nabla^2\Phi + f^2 + A^2_n + B^2_n - 2\nabla f.\nabla\psi^{(n)}} \,. \qquad (2.2)$$

Writing

$$F(\psi^{(n)}) - F(\psi^{(n-1)}) = \frac{F^2(\psi^{(n)}) - F^2(\psi^{(n-1)})}{F(\psi^{(n)}) + F(\psi^{(n-1)})},$$

we find, neglecting the term $2\nabla f.\nabla\varepsilon^{(n)}$,

$$\nabla^2\varepsilon^{(n+1)} = \frac{A^2_n - A^2_{n-1} + B^2_n - B^2_{n-1}}{2f + \nabla^2(\psi^{(n+1)} + \psi^{(n)})}, \qquad (2.3)$$

where

$$A_n^2 - A_{n-1}^2 = ((\psi^{(n)} + \psi^{(n-1)})_{xx} - (\psi^{(n)} + \psi^{(n-1)})_{yy}) \quad (\varepsilon_{xx}^{(n)} - \varepsilon_{yy}^{(n)}) \quad ,$$

$$B_n^2 - B_{n-1}^2 = 4 (\psi^{(n)} + \psi^{(n-1)})_{xy} \quad \varepsilon_{xy}^{(n)} \quad \text{and}$$

$$\nabla^2 (\psi^{(n+1)} + \psi^{(n)}) = \nabla^2(\psi^{(n)} + \psi^{(n-1)}) + \nabla^2(\varepsilon^{(n+1)} + \varepsilon^{(n)}) \; .$$

With $\bar{\psi} = \psi^{(n)} + \psi^{(n-1)}$ equation (2.3) passes into

$$\nabla^2 \varepsilon^{(n+1)} = \frac{(\bar{\psi}_{xx} - \bar{\psi}_{yy}) \; (\varepsilon_{xx}^{(n)} - \varepsilon_{yy}^{(n)}) + 4 \bar{\psi}_{xy} \; \varepsilon_{xy}^{(n)}}{2f + \nabla^2 \bar{\psi} + \nabla^2(\varepsilon^{(n+1)} + \varepsilon^{(n)})} \quad . \tag{2.4}$$

We remark that $\varepsilon^{(i)} = 0$, $(i=1,2, \ldots)$ on the boundary. For simplicity we consider equation (2.4) on a square $(0 \leq x,y \leq \pi)$ treating the functions $\bar{\psi}_{xx}$, $\bar{\psi}_{yy}$ and $\bar{\psi}_{xy}$ as constants. Further we linearize equation (2.4) by neglecting the terms $\nabla^2\varepsilon^{(n+1)}$ and $\nabla^2\varepsilon^{(n)}$ in the right-hand side so that we have instead of (2.4)

$$\nabla^2 \varepsilon^{(n+1)} = \frac{(\bar{\psi}_{xx} - \bar{\psi}_{yy}) \; (\varepsilon_{xx}^{(n)} - \varepsilon_{yy}^{(n)}) + 4 \bar{\psi}_{xy} \, \varepsilon_{xy}^{(n)}}{2f + \nabla^2 \bar{\psi}} \quad . \tag{2.5}$$

In view of the foregoing assumptions we may expand $\varepsilon^{(n)}$ and $\varepsilon^{(n+1)}$ in the double Fourier series

$$\varepsilon^{(n)} = \sum_{k,m=-\infty}^{\infty} A_{k,m}^{(n)} \; e^{\,i(kx+my)} \quad ,$$

$$\varepsilon^{(n+1)} = \sum_{k,m=-\infty}^{\infty} A_{k,m}^{(n+1)} \; e^{\,i(kx+my)} \; . \tag{2.6}$$

Substitution of (2.6) into (2.5) gives the following relation between $A_{k,m}^{(n+1)}$ and $A_{k,m}^{(n)}$ $(k,m \neq 0)$

$$A_{k,m}^{(n+1)} = \frac{(\bar{\psi}_{xx} - \bar{\psi}_{yy})\ (k^2 - m^2) + 4\,km\ \bar{\psi}_{xy}}{(k^2 + m^2)\ (2f + \nabla^2\bar{\psi})}\ A_{k,m}^{(n)} \quad . \tag{2.7}$$

From (1.4) we have

$$f + \bar{\psi}_{xx} = \varepsilon_1 > 0 \ , \qquad f + \bar{\psi}_{yy} = \varepsilon_2 > 0 \ . \tag{2.8}$$

In order that $\varepsilon^{(n)}$ converges towards zero the inequality

$$\left| \frac{(\bar{\psi}_{xx} - \bar{\psi}_{yy})\ (k^2 - m^2) + 4\,km\ \bar{\psi}_{xy}}{(k^2 + m^2)\ (2f + \nabla^2\bar{\psi})} \right| < 1 \tag{2.9}$$

must hold for every $k,m \neq 0$. Using (2.8) the amplification factor in (2.7) may be written as

$$\frac{k^2\varepsilon_1 + m^2\varepsilon_2 - k^2\varepsilon_2 - m^2\varepsilon_1 + 4\,km\,\bar{\psi}_{xy}}{k^2\varepsilon_1 + m^2\varepsilon_2 + k^2\varepsilon_2 + m^2\varepsilon_1} \ . \tag{2.10}$$

If the successive approximate solutions $\psi^{(i)}$ (i=0,1,2, ...) are required to satisfy the ellipticity condition (1.2) it follows that

$$|\,\bar{\psi}_{xy}\,| < \sqrt{\varepsilon_1\varepsilon_2} \ . \tag{2.11}$$

Writing

$$k^2\varepsilon_1 + m^2\varepsilon_2 = (\,|k|\,\sqrt{\varepsilon_1} - |m|\,\sqrt{\varepsilon_2})^2 + 2\,|km|\,\sqrt{\varepsilon_1\varepsilon_2} \ ,$$

$$k^2\varepsilon_2 + m^2\varepsilon_1 = (\,|k|\,\sqrt{\varepsilon_2} - |m|\,\sqrt{\varepsilon_1})^2 + 2\,|km|\,\sqrt{\varepsilon_1\varepsilon_2} \ ,$$

the amplification factor (2.10) reads

$$\frac{(|k|\sqrt{\varepsilon_1}-|m|\sqrt{\varepsilon_2})^2-(|k|\sqrt{\varepsilon_2}-|m|\sqrt{\varepsilon_1})^2+4\,km\,\bar{\psi}_{xy}}{(|k|\sqrt{\varepsilon_1}-|m|\sqrt{\varepsilon_2})^2+(|k|\sqrt{\varepsilon_2}-|m|\sqrt{\varepsilon_1})^2+4|km|\sqrt{\varepsilon_1\varepsilon_2}}\ ,$$

so that, in view of (2.11), condition (2.9) holds. To summarize, the functions $\psi^{(i)}$ (i=0,1,2, ...) which satisfy (2.1) will converge towards the solution ψ of (1.6) if a sufficiently accurate initial approximation $\psi^{(0)}$ of this solution is available, provided all the approximate functions satisfy conditions (1.2) and (1.4).

If the approximate solutions do not satisfy the ellipticity condition (1.2) the absolute value of the quotient (2.10) may be not very much different from one, causing an oscillation in the solution of the numerical procedure (2.1), which may result in a very slow convergence (or divergence). To suppress this oscillation we solve, in stead of (2.1), the pair of equations

$$\nabla^2\phi^{(n+1)} = -f + F(\psi^{(n)})\ ,$$

$$\psi^{(n+1)} = \phi^{(n+1)} + \omega\,(\psi^{(n)} - \phi^{(n+1)})\ ,$$

where $F(\psi^{(n)})$ is given by (2.2). Elimination of $\phi^{(n+1)}$ gives

$$\nabla^2\psi^{(n+1)} = \omega\nabla^2\psi^{(n)} + (1-\omega)\,(-f + F(\psi^{(n)}))\ . \qquad (2.12)$$

If we write $\omega = \alpha/(1+\alpha)$ equation (2.12) reads

$$(1+\alpha)\,\nabla^2\psi^{(n+1)} = \alpha\nabla^2\psi^{(n)} - f + F(\psi^{(n)})\ ,$$

which is the form of the modified iterative method of Shuman, used by Paegle and Tomlinson (1975, eq. (11), p. 531).

Analogous to the treatment of (2.1) we can analyze the convergence of the iterative method (2.12). From this analysis it appears that also this method may diverge if the ellipticity condition is violated too much by successive approximate solutions. In the experiments described in section 4 we shall try to find out whether the convergence behaviour of the iterative method (2.1) is really dominated by the ellipticity condition as we pointed out in this section.

3. The computational method.

The transformation of equation (1.1), being written in tangent plane coordinates, onto the south polar stereographic projection (the projection plane passes through the circle of 60 N latitude) is given by

$$m^2 f \nabla^2\psi + 2m^4 (\psi_{xx} \psi_{yy} - \psi_{xy}^2) + m^2 \nabla f.\nabla\psi - m^2 \nabla^2\Phi = 0 , \qquad (3.1)$$

where m (ϕ) is the map factor

$$m(\phi) = (1+\sin \pi/3) / (1+\sin\phi) ,$$

and ϕ is the latitude. Analogous to (3.1) equation (1.6) reads

$$m^2 \nabla^2\psi = -f + \sqrt{f^2 + m^4 (A^2 + B^2) - 2m^2 \nabla f.\nabla\psi + 2m^2 \nabla^2\Phi}. \qquad (3.2)$$

If Γ denotes the boundary of the region where (3.2) applies, then ψ (x,y) is to satisfy the Dirichlet boundary condition

$$\psi (x,y) = \Phi(x,y) / f, \qquad (x,y) \in \Gamma.$$

We now impose a uniform square grid (x_i, y_j) on this region, with mesh side d. The boundary consists of rectangular segments along horizontal and vertical mesh lines so that an interior point is surrounded by eight mesh points, each of which being an interior or a boundary mesh point. Using the notation $\psi(i,j) = \psi(x_i, y_j)$, $f(i,j) = f(x_i,y_j)$ and $m(i,j) = m(x_i,y_j)$ partial derivatives are replaced by the following usual finite difference approximations at an interior point (i,j)

$$\psi_{xx} : (\psi(i-1,j) + \psi(i+1,j) - 2\psi(i,j)) / d^2,$$
$$\psi_{yy} : (\psi(i,j-1) + \psi(i,j+1) - 2\psi(i,j)) / d^2,$$
$$\psi_{xy} : (\psi(i+1,j+1) + \psi(i-1,j-1) - \psi(i+1,j-1) - \psi(i-1,j+1)) / 4d^2,$$
$$2\nabla f.\nabla\psi : ((f(i+1,j) - f(i-1,j))(\psi(i+1,j) - \psi(i-1,j)) + (f(i,j+1) - f(i,j-1))(\psi(i,j+1) - \psi(i,j-1)))/ 2d^2 = \gamma/d^2$$

so that

$$\nabla^2\psi \quad : \; (\psi(i+1,j) + \psi(i-1,j) + \psi(i,j-1) + \psi(i,j+1) - 4\,\psi(i,j))/\,d^2 = D\,\psi(i,j)\,/\,d^2,$$

$$A^2 + B^2 \; : \; (\psi(i-1,j) + \psi(i+1,j) - \psi(i,j-1) - \psi(i,j+1))^2 + \tfrac{1}{4}(\psi(i+1,j+1) + \psi(i-1,j-1) - \psi(i+1,j-1) - \psi(i-1,j+1))^2\,/\,d^4 = (\alpha^2+\beta^2)\,/\,d^4,$$

and equation (3.2) becomes

$$m^2\,D\psi\,/\,d^2 = -f + \sqrt{f^2 + m^4\,(\alpha^2+\beta^2)\,/\,d^4 - m^2\gamma\,/\,d^2 + 2m^2\,D\Phi\,/\,d^2}\;. \qquad (3.3)$$

Equation (3.3) is solved by the iteration proces

$$D\,\psi^{(n+1)} = -f\,d^2/m^2 + \sqrt{(f\,d^2/m^2)^2 + \alpha_n^2 + \beta_n^2 - \gamma_n\;d^2/m^2 + 2\;D\,\Phi d^2/m^2}, \qquad (3.4)$$

where the quantities α_n, β_n and γ_n are calculated using the n-th iterate $\psi^{(n)}$. As an initial guess we take $\psi^{(0)} = \Phi/\bar{f}$, where $\bar{f}$ is the average Coriolisparameter.

Analogous to condition (1.2) of section 1, the condition that equation (3.1) be elliptic is given by

$$(f/m^2 + 2\,\psi_{yy})\,(f/m^2 + 2\,\psi_{xx}) - 4\,\psi_{xy}^2 \quad > 0 \qquad (3.5)$$

or equivalently

$$2\,\nabla^2\Phi + f^2/m^2 - 2\,\nabla f.\nabla\psi \quad > 0. \qquad (3.6)$$

Of course conditions (3.5) and (3.6) are equivalent if ψ is a solution of (3.1) and they are not equivalent for functions $\psi^{(n)}$ which are approximate solutions of (3.1) obtained by an iterative method such as (2.1).

In the following we shall consider the finite difference version of (3.5) as the ellipticity condition of our problem and we shall show in the next section that this condition plays an important rôle with respect to the convergence of the iterative procedure (3.4). To be more specific it will be shown that the iterative method (3.4) will be convergent if the subsequent approximate solutions $\psi^{(n)}$ satisfy the finite difference version of (3.5). Otherwise very slow convergence or even divergence may occur.

In practice however we shall be concerned with condition (3.6). The better (3.6) is satisfied by an approximate solution $\psi^{(n)}$ of (3.4), the better (3.5) will be satisfied. In the experiments of the next section we use condition (3.6) in three different ways.

(a) At those points where $2\,D\,\Phi + f^2 d^2/m^2 < 0$ the geopotential is modified so that $2\,D\,\Phi = -f^2 d^2/m^2$. During the iteration proces (3.4) we substitute zero for the expression

$$(f\,d^2/m^2)^2 + \alpha_n^2 + \beta_n^2 - \gamma_n\,d^2/m^2 + 2\,D\,\Phi\,d^2/m^2$$

when it is negative.

(b) The first iteration of (3.4) the geopotential is changed at those points where $2\,D\,\Phi + f^2 d^2/m^2 < 0$ according to (a). During further iteration the geopotential is modified again at those points where

$$2\,D\,\Phi + f^2 d^2/m^2 - \gamma_n < 0$$

so that

$$2\,D\,\Phi = \gamma_n - f^2 d^2/m^2.$$

This will give only small changes in Φ

(c) We now proceed in a manner given by Shuman (1957 a):
"The field of $Z = 2D\Phi + f^2d^2/m^2$ is scanned with a test for negative values. When a negative value of Z is encountered the values at the surrounding nearest four points are reduced by $\frac{1}{4}$ of the magnitude of Z at the central point and the value of Z at the central point is increased to zero. Boundary values are excepted from change". This operation is applied 20 times (see section 4). With respect to the expression

$$(fd^2/m^2)^2 + \alpha_n^2 + \beta_n^2 - \gamma_n\, d^2/m^2 + 2D\Phi\, d^2/m^2$$

we refer to (a).

In the case where the numerical solutions start to oscillate we may solve the pair of equations

$$D\phi^{(n+1)} = -f\,d^2/m^2 + \sqrt{(f\,d^2/m^2)^2 + \alpha_n^2 + \beta_n^2 - \gamma_n\, d^2/m^2 + 2D\Phi\, d^2/m^2},$$

$$\psi^{(n+1)} = \phi^{(n+1)} + \omega\,(\psi^{(n)} - \phi^{(n+1)}),\ 0 < \omega < 1. \qquad (3.7)$$

Iteration is stopped when a function $\psi^{(n)}$ is obtained which makes the maximum of the absolute values of the residuals with respect to equation (3.3) less than a prescribed quantity. Here a residual is defined as the difference between the right and left member of (3.3). Each iteration step we have to solve a Poisson equation. If this is done by an iterative method (as is the case in this report) the required accuracy must be chosen in accordance with this stop criterion (see also section 4).

4. Experiments.

The tabulated geopotential is smoothed by the pair of one-dimensional smoothers (Shuman, 1957 b)

$$\Phi(i,j) = 0.72764\,\Phi(i,j) + 0.22049\,(\Phi(i-1,j) + \Phi(i+1,j)) - 0.11318\,(\Phi(i-2,j) + \Phi(i+2,j)) + 0.02886\,(\Phi(i-3,j) + \Phi(i+3,j))$$

and

$$\Phi(i,j) = 0.72764\,\Phi(i,j) + 0.22049\,(\Phi(i,j-1) + \Phi(i,j+1)) - 0.11318\,(\Phi(i,j-2) + \Phi(i,j+2)) + 0.02886\,(\Phi(i,j-3) + \Phi(i,j+3)).$$

These smoothers are passed successively six times over the geopotential field. In the following $\overline{F}(\psi^{(n)})$ stands for the square root in the right member of (3.4). Using the stop criterion

$$\max_{i,j} \left| D\,\psi^{(n)} + f\,d^2/m^2 - \overline{F}(\tilde{\psi}^{(n)}) \right| < \delta, \qquad (4.1)$$

for the iterative method (3.4), where $\tilde{\psi}^{(n)}$ is the approximate solution of

$$D\,\psi^{(n)} = -\,f\,d^2/m^2 + \overline{F}(\psi^{(n-1)}) \qquad (4.2)$$

(or $D\psi^{(n)} = \omega D\psi^{(n-1)} + (1-\omega)\,(-\,f\,d^2/m^2 + \overline{F}(\psi^{(n-1)}))$ when solving (3.7)) this approximate solution must be computed to within an accuracy so that in equality (4.1) can be satisfied. It appears to be sufficient to solve the discrete Poisson equation (4.2) with an accuracy of

$$\max_{i,j} \left| D\tilde{\psi}^{(n)} - D\psi^{(n)} \right| < 10^{-q} \max_{i,j} \left| D\psi^{(n-1)} + f\,d^2/m^2 - \overline{F}(\psi^{(n-1)}) \right|$$

with $q = 1$. In the following experiments $\delta = 10^3$ and $d = 375.10^3$

so that

$$\max_{i,j} \left| m^2 D \psi^{(n)} / d^2 + f - m^2 \overline{F} (\tilde{\psi}^{(n)}) / d^2 \right| < \frac{10^{-7}}{(3.75)^2} \max_{i,j} m^2(i,j).$$

As noted in the previous section, it is the intention of the experiments to show that convergence will be slower if the ellipticity condition (3.5) is more negative at a larger number of points. The balance equation is solved on an octagonal region with 1624 interior mesh points (see Fig. 1).

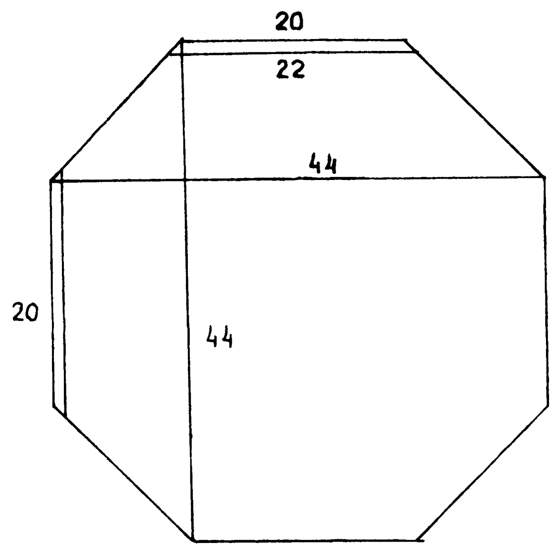

Fig. 1. The number of interior mesh points is indicated for horizontal and vertical mesh lines.

The 12-hour 500-mb geopotential fields used in this report are taken from the period 5-1-1978 through 8-1-1978. The results presented below are representative for other experiments that we have done.

	case (b)	case (a)	case (c)
010500	16(4)	17(12)	32(95)
12	17(1)	17(11)	* (74)
0600	19(4)	19(16)	25(88)
12	17(3)	18(10)	29(46)
0700	24(1)	42(11)	48(65)
12	15(1)	15(11)	26(45)
0800	27(0)	28(18)	36(55)
12	17(0)	17(5)	33(49)

Table 1. For explanation of case (a), (b), (c), see section 3.

In Table 1 the number of iterations is given needed to satisfy the stop criterion (4.1).• Between brackets we give the number of points at which the ellipticity condition was negative during the last iteration. For instance in case (b) of 8-1-1978 at 00 GMT the ellipticity condition was very negative at a few points during many preceding iterations, resulting in a relatively slow convergence. It was not possible to reduce the number of iterations in case (c) of 5-1-1978 12 GMT by applying the iterative method (3.7).

• (* means more than 50 iterations)

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