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J. Oerlemans

On the stability of
Northern Hemisphere continental ice sheets.

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Koninklijk Nederlands Meteorologisch Instituut,
Meteorologisch Onderzoek,
Postbus 201,
3730 AE De Bilt,
Nederland.

U.D.C.: 551.583.3 :
551.324.24

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Summary and conclusions

In this study we discuss the behaviour of large continental ice sheets under various climatic conditions. We consider ice sheets that behave perfectly plastic, i.e. their shape is parabolic and depends only on their size, no matter how the mass balance varies in space. In this case it is possible to study the feedback between ice-sheet height and mass balance with analytical tools, as was done first by Weertman (1961a).

The analysis suggests that three characteristic situations exist ($\simeq$ means order of magnitude):

(i) Warm climatic conditions: one stable equilibrium solution exists, namely $L=0$ (L is ice-sheet size in meridional direction).

(ii) Moderate climate conditions: three equilibrium solutions exist: $L=0$ (stable), $L \simeq 200$ km (unstable) and $L \simeq 2000$ km (stable).

(iii) Cold climatic conditions: only one equilibrium solution exists: $L \simeq 3000$ km, which is stable.

Apparently, the ice sheet shows hysteresis. This implies that ice sheets probably respond strongly nonlinear to quasi-periodic insolation variations.

Analysis of deep-sea core V12-122 (Imbrie et al., 1974) shows uni-modal distributions of parameters directly related to sea-surface temperature, and bimodal distributions of oxygen isotope ratios, which are related to ice volume on earth. This fact strongly supports the theoretical results of this study.

1. Introduction

Of the various theories put forward to explain the quaternary ice ages, the large sensitivity of ice caps to external (= atmospheric) conditions has never got much attention. Bodvarsson (1955) was the first to mention explicitly that the coupling between surface elevation and annual ice-mass balance substantially enhances the sensitivity of continental ice sheets. Weertman (1961a, 1961b, 1964, 1976) investigated this point in a quantitative, but crude, way. He considered two-dimensional ice sheets (height and latitude) and parameterized the mass budget by prescribing a snow line that separates regions of constant ablation and constant accumulation. Weertman found that the equilibrium size was very sensitive to the values of ablation and accumulation.

In this study we will carry out an analysis similar to Weertman's studies, but we will use a more realistic prescription of the mass balance.

2. Mass budget and ice-sheet profile

We start our discussion by formulating the annual ice-mass balance G. Observations have shown that G increases with height and latitude (Charlesworth, 1957). For the height dependence we write

$$G = a(h-E) - b(h-E)^2 \qquad (1)$$

where h is height above sea level and E is the height of the equilibrium line (defined by $G = 0$). The constants a and b are positive, so eq. (1) describes a parabola with the top above the equilibrium line. Requiring that this top is at $h-E=1500$ m, and the corresponding mass balance 0.5 m/y, gives $a=0.732 \cdot 10^{-3}\ y^{-1}$ and $b=0.268 \cdot 10^{-6}\ (my)^{-1}$. If $h-E > 1500$ m, we set $G=0.5$ m/y, see Fig. 1.

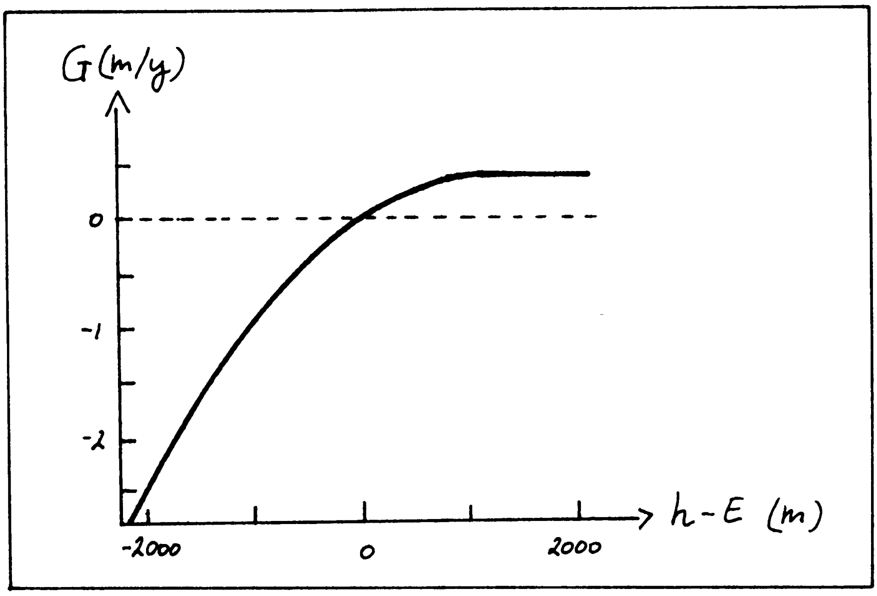

Fig. 1. Height-dependence of the annual mass balance G.

We now include the latitudinal dependence of G by prescribing E as

$$E = \chi x + \Theta \qquad (2)$$

Here, x is the distance along a meridian in the southward direction from 70° N. So x measures approximately the distance to the polar sea. χ is the slope of the lines of equal mass balance. The point where the equilibrium line intersects sea level is given by $P = -\Theta/\chi$ (climate reference-point). Fig. 2 gives an example of the distribution of G described by eqs. (1) and (2). χ has been set at 0.001. The climate reference-point lies 1000 km north of 70° N; this corresponds to conditions slightly warmer than at present.

The shape of a continental ice sheet is determined by its mechanics, but the direct driving force is of course its mass balance. An ice sheet grows/shrinks when the mean mass balance is positive/negative. We thus may learn a lot if we investigate how the mass balance of an ice sheet depends on its size. In order to find the total mass balance, we have to integrate G(h,x) along the surface of the ice sheet. So if we want to derive a functional relationship between the total mass balance and the

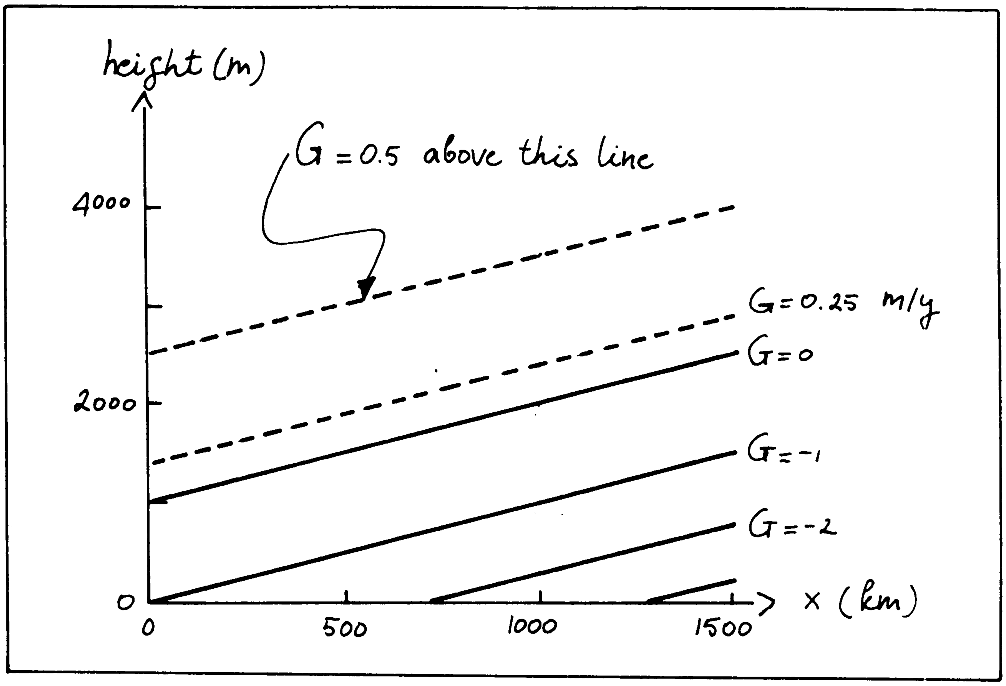

Fig. 2. Parameterization of the mass balance G.
Constants used: $a = 0.7321 * 10^{-3}\ y^{-1}$;
$b = 0.268 * 10^{-6}\ (my)^{-1}$; $\chi = 10^{-3}$; $\Theta = 1000$.

ice-sheet size (or volume), we need a unique relation between ice-sheet profile and size that does not depend on the variation of G over the sheet. Such a unique relation exists if we assume that the ice sheet behaves perfectly plastic. For a discussion on perfect plasticity, see Paterson (1969) .

The profile of a perfectly plastic ice sheet on a horizontal bedrock is given by

$$H(x) = \sigma \left[\tfrac{1}{2} L - | x - \tfrac{1}{2} L | \right]^{\frac{1}{2}} \qquad (3)$$

Here H(x) is the height of the sheet, L its size and σ a constant that depends on the yield stress of ice and determines the height to width ratio. In this study we use $\sigma = 2.5\ m^{\frac{1}{2}}$, which means that a sheet with L = 2000 km has a maximum height of 2500 m. Eq. (3) follows directly from the requirement that the shear stress at the bottom equals the (constant) yield stress everywhere. Fig. 3 shows a perfectly plastic ice sheet. In the case of equilibrium, there is no horizontal flux of ice through the centre of the sheet. This implies that we can find the size of the sheet by making up the mass balance over the southern half.

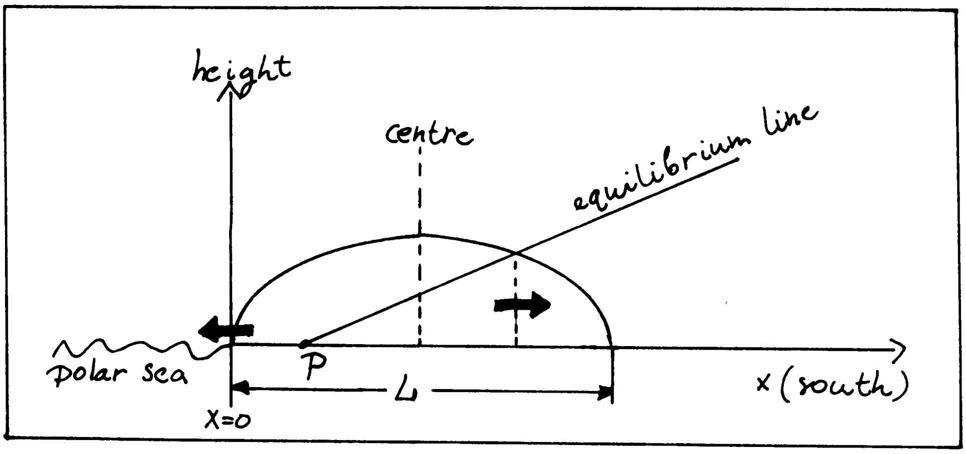

Fig. 3. A perfectly plastic ice sheet of size L. Locations where the ice-mass discharge is largest are indicated with heavy arrows. There is no mass transport through the centre of the sheet.

Inserting (2) in (1) yields for the mass balance:

$$G(x,h) = ah - a\chi x - a\Theta - bh^2 - b\chi^2 x^2 - b\Theta^2 - 2b\chi\Theta x + 2b\chi hx + 2b\Theta h \; . \qquad (4)$$

If we now substitute the height of the ice sheet H(x) for h and integrate from $x = \frac{1}{2}L$ to $x = L$, we obtain for the mass balance averaged over the southern half of the sheet:

$$G^*(L) = \frac{2}{L}\int_{\frac{1}{2}L}^{L} G(x)dx = A_1 + A_2 L^{\frac{1}{2}} + A_3 L + A_4 L^{3/2} + A_5 L^2 \quad , \qquad (5)$$

where
$$A_1 = -\Theta(a + b\Theta)$$
$$A_2 = .471\,\sigma(a + 2b\Theta)$$
$$A_3 = -0.25\,b\sigma^2 - 0.75\,\chi(a + 2b\Theta)$$
$$A_4 = 0.666\,b\sigma\chi$$
$$A_5 = -0.583\,b\chi^2$$

Evaluation of (5) for a given set of constants allows us to determine the equilibrium size(s) of an ice sheet: we simply have to require $G^* = 0$. Whether an equilibrium solution is stable depends on $\partial G^*/\partial L$. If $\partial G^*/\partial L > 0$, the equilibrium solution is unstable; if $\partial G^*/\partial L < 0$, it is stable.

Although the expression for $G^*(L)$ looks rather complicated, it is not, as we will see in the subsequent sections.

3. Mass budget and equilibrium solutions

In the evaluation of (5), difficulties may arise if h-E becomes larger than 1500 m for any $x > \frac{1}{2}L$. This is a consequence of the fact that we did not include the conditions G = 0.5 m/y if h-E > 1500 m in the computation of the total mass balance. In practice however, this plays a role for extremely large ice sheets only (L > 3000 km or so, depending to some extent on the value of χ). In the following, the values of a, b and σ are equal to those already mentioned, unless stated otherwise.

Fig. 4 shows how G^* depends on the ice-sheet size. It rapidly increases with L, reaches a maximum and then decreases steadily. This type of behaviour was found for all kind of sets of constants (within their realistic range), so the conclusions to be drawn from Fig. 4 have general value.

We first note that the influence of Θ is very clear: changing Θ, which may be interpreted as varying climatic conditions, shifts the curve up and down while its shape is hardly affected. Since equilibrium solutions are found by

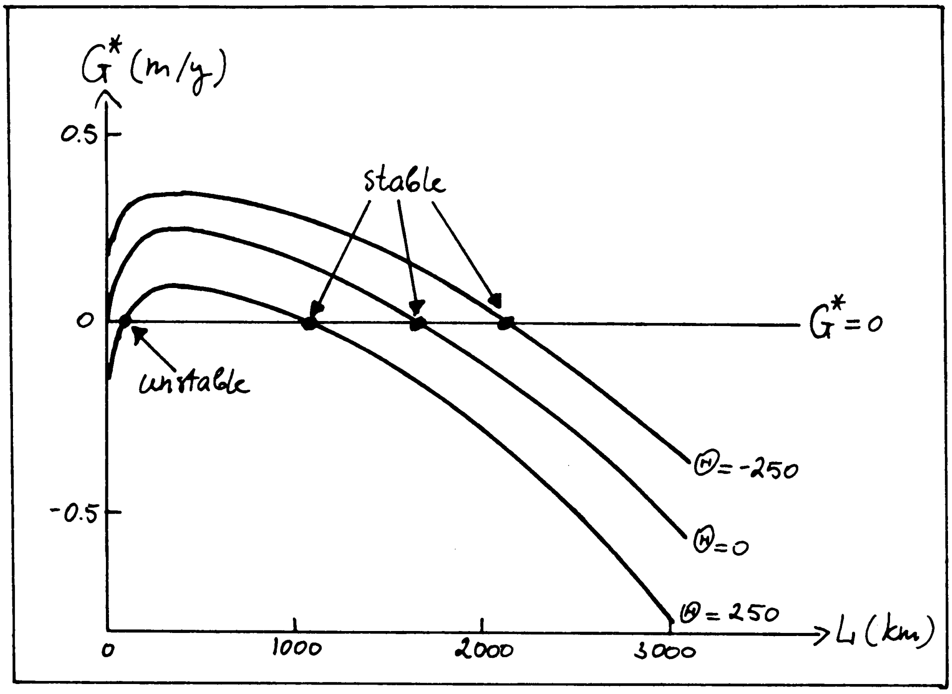

Fig. 4. G^* as a function of ice-sheet size L for three values of Θ. Equilibrium solutions are given by the intersections of the curve with $G^* = 0$ (dashed line).

equating G^* to zero, the maximum number of solutions is two. In the case that $\Theta = -250$ m, the climatic reference-point P (the point where the equilibrium intersects sea level) is situated 250 km from the polar sea (in southward direction). Only one equilibrium solution exists: L = 2200 km. This solution is stable because $\partial G^*/\partial L < 0$ here. If $\Theta = 250$ m, i.e. if the climate reference-point P is situated in the polar sea, Fig. 4 reveals two solutions: a small sheet which is unstable, and a large sheet which is stable.

At this point we note that for $\Theta > 0$ (corresponding to $P < 0$), $L = 0$ is a stable solution, which is not revealed by

Fig. 4. It directly follows from the fact that in this case G^* is always negative for $x \geqslant 0$.

Fig. 4 also suggests that some Θ_0 exists, such that for $\Theta > \Theta_0$, $L = 0$ is the only solution. In this case the top of the G^*-curve is below the line $G^* = 0$.

Summarizing, we may distinguish three cases:

(i) $\Theta \leqslant \Theta_0$: the only solution is $L = 0$, which is stable;

(ii) $\Theta_0 < \Theta \leqslant 0$: three solutions exist: $L = 0$ (stable), a small sheet (unstable) and a large sheet (stable);

(iii) $\Theta < 0$: only one solution exists: a large ice sheet, which is stable.

Only in case (ii) two <u>stable</u> solutions exist - here it depends on the initial state to which solution the ice sheet will grow. In other words, the present analysis shows that we have to do with hysteresis. This is qualitatively shown in Fig. 5. The cases discussed above are indicated.

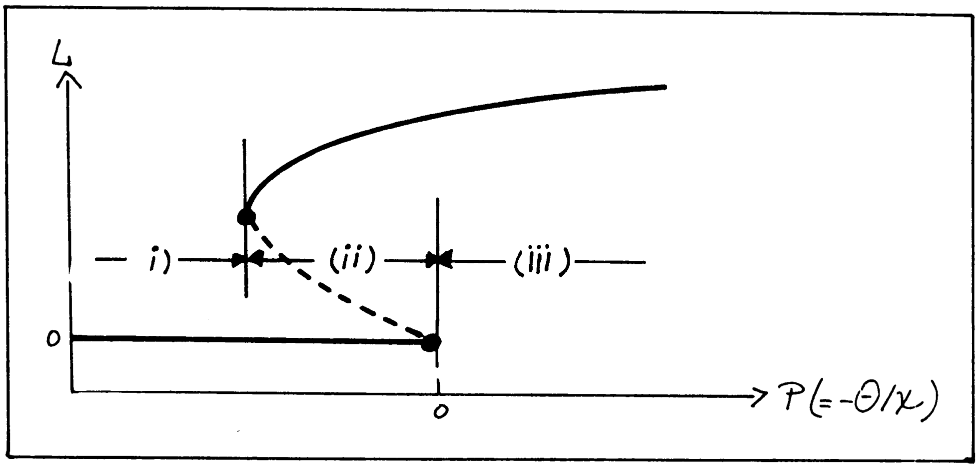

Fig. 5. Illustration of the analysis of equilibrium solutions. Stable solutions are indicated with solid lines. The dashed line indicates unstable solutions.

In order to arrive at a more quantitative solution diagram, we could compute the zero's of $G^*(L)$ for various values of Θ. Because $G^*(L)$ is in fact a polynomal of fourth order, this has to be done with a numerical method. On the other hand, the fact that (for realistic constants) not more than two zero's occur suggests that a quadratic equation might be sufficient to determine the solutions. We turn to this point in the next section.

4. Simplification of $G^*(L)$

From (5) we see that G^* would be quadratic in L if b would be zero, i.e. if the mass balance is prescribed as a linear function of height. Some sample calculations were carried out to see whether this makes much difference. It appeared that the shape of the G^*-curve is not affected, but the points where $G^* = 0$ shift somewhat. In view of the uncertainties in other parameters, in particular in χ, it seems justified to prescribe G linearly in height.

We thus rewrite G as

$$G = \alpha(x - P) + \beta h \quad , \tag{6}$$

where P is the climate reference-point again, and α and β are constants (orders of magnitude: $\alpha \simeq -10^{-6}\ y^{-1}$, $\beta \simeq 10^{-3}\ y^{-1}$). The mass balance averaged over the southern half of the sheet now becomes

$$G^*(L) = B_1 + B_2 L^{\frac{1}{2}} + B_3 L \quad , \tag{7}$$

where
$$B_1 = -\alpha P$$
$$B_2 = 0.4714\ \beta\sigma$$
$$B_3 = 0.75\ \alpha$$

We first observe that variations in P shift the G^*-curve up and down without affecting its shape. Therefore, we consider the case P = 0. Fig. 6 shows $G^*(L)$ for three values of α. Apparently, the stable equilibrium solution is very sensitive

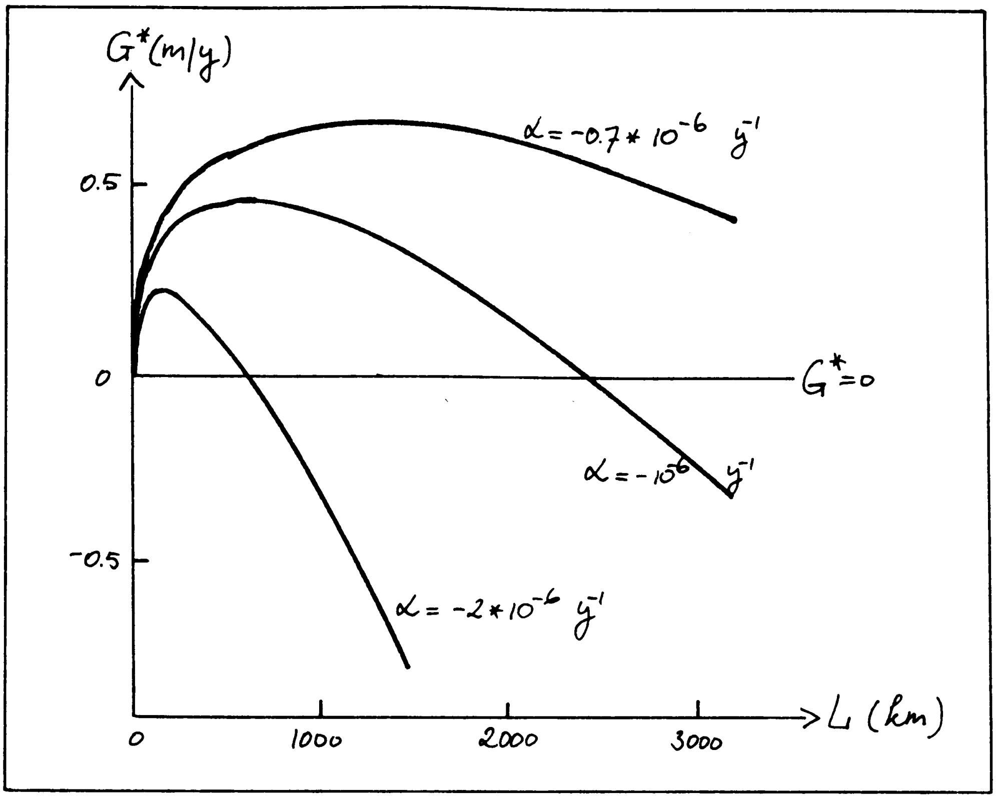

Fig. 6. G*(L) for three values of α. P has been set to zero.

to α (determining the slope of the equilibrium line). If this slope is small, stable equilibrium solutions become very large.

5. Computation of the solution diagram

The equilibrium solutions are found by equating G* to zero. Thus from (7) we have

$$B_1 + B_2 L^{\frac{1}{2}} + B_3 L = 0 \tag{8}$$

After squaring, this equation can be solved to give

$$L_{1,2} = \frac{B_2^2 - 2B_1B_3 \pm B_2\sqrt{B_2^2 - 4B_1B_3}}{2B_3^2} \, . \tag{9}$$

No real solutions of $L_{1,2}$ exist if $B_2^2 - 4B_1B_3 < 0$. In terms of α, β, σ and **P** this condition becomes

$$P < -0.741\,\sigma^2\beta^2/\alpha^2 . \qquad (10)$$

We call the right-hand side of (10) P_o, the critical point. Note that P_o is negative, i.e. it lies in the polar sea. If $P < P_o$, thus if P lies north of the critical point, (8) gives no solutions. If $P > P_o$, (8) gives two solutions. It may be shown that those solutions are always positive. From Figs. 4 and 6 we know that if $P > 0$, only one solution exists. The reason that (9) gives two solutions in this region simply is the fact that we squared (8).

We are able now to construct a solution diagram. $L_{1,2}$ are easily computed from (9) and furthermore we know that $L = 0$ is a stable solution if $P < 0$. Fig. 7 shows solution diagrams for three values of α. Stable solutions are given with solid lines, unstable solutions with dashed lines. Critical points are indicated with black spots.

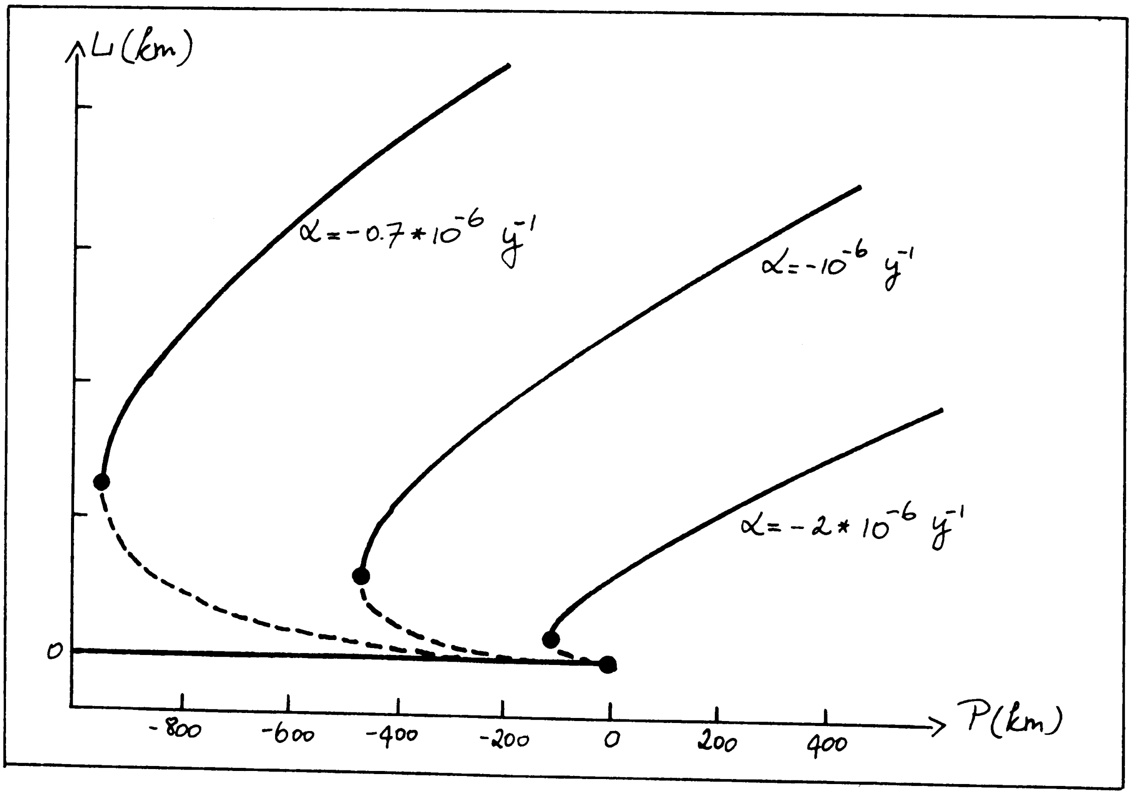

Fig. 7. Solution diagram for perfectly plastic ice sheets. Values of constants: $\sigma = 2.5\ m^{\frac{1}{2}}$, $\beta = 10^{-3}\ y^{-1}$.

From Fig. 7 we see that ice sheets of considerable size are possible even if P is situated a few hundred kilomaters north of the northern coast of the continent. If we imagine that climatic variations shift P north- and southwards in a smooth way, the response of ice sheets may be quite complicated. Or, in other words, the response of ice sheets to regular insolation variations may have a strong nonlinear character. In the following section we will make some inferences on this point.

6. Inferences.

The fact that the ice-sheet behaviour is characterized by hysteresis in the range of climatic conditions that prevailed during the last million years or so is very important. It results from the fact that the major continents in the Northern Hemisphere are bounded in the north by the polar sea. Weertman was the first to recognize that this configuration gives raise to the appearence of two preferred states: no ice sheet and a large ice sheet. Thus, if the time scale of ice-sheet growth and decay is not much larger than that of the forcing of the climate system (presumably variations in the earth's orbit), a frequency distribution of ice volume on earth should have a somewhat bimodal shape. With the palaeoclimatic data available at presnt, this point can be verified.

Fig. 8 shows frequency distributions of wintertime sea-surface temperature in the Carribean (uppper part) and of δO^{18}, which measures the total ice volume on earth. Variations in this volume are mainly determined by variations in the Fennoskandian and Laurentide sheets. Without doubt, those distributions show significant differences.

The sea-surface temperature distribution resembles a Gaussian distribution. This may be explained by the roughly linear response of sea-surface temperature to insolation

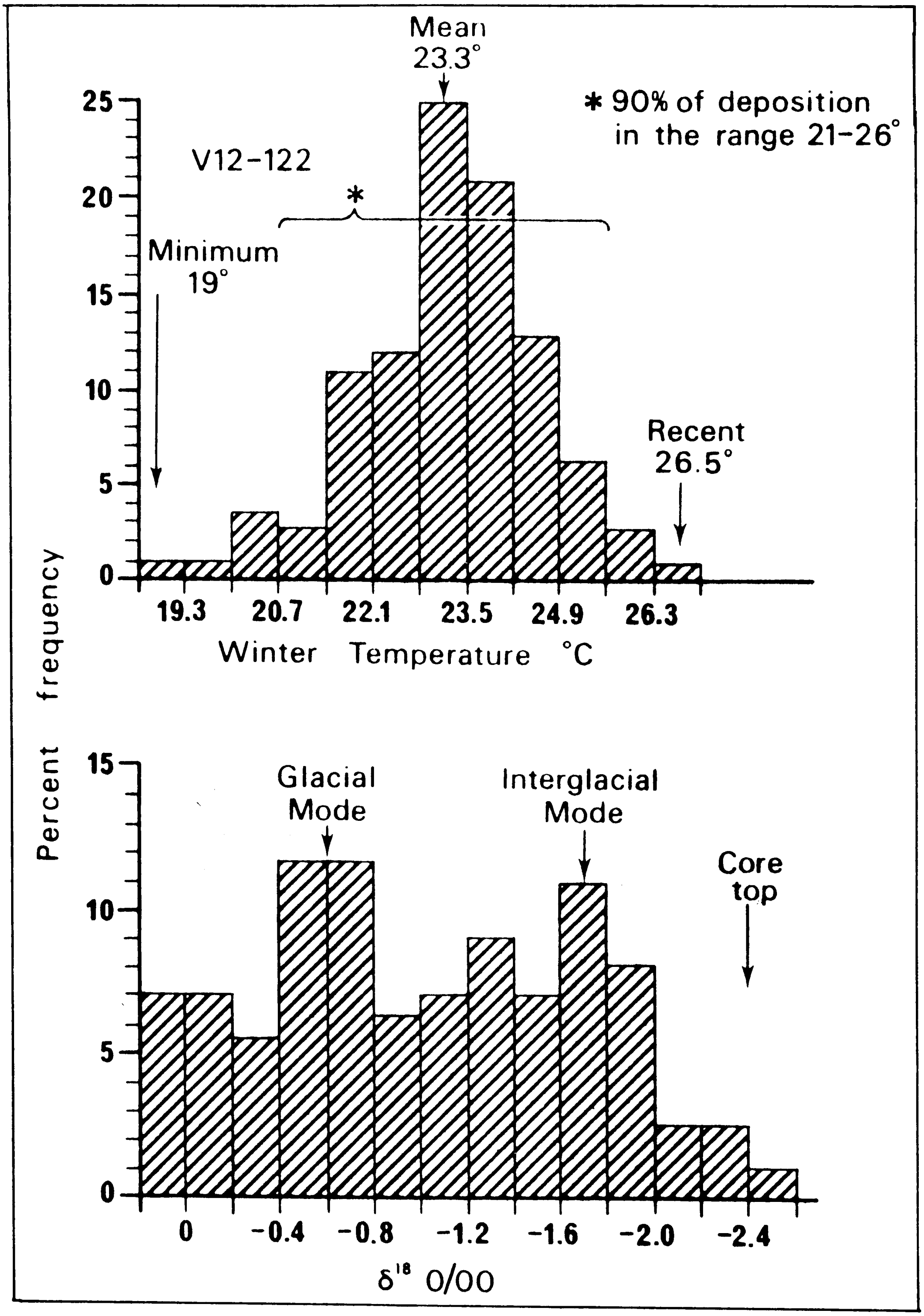

Fig. 8. Histograms computed from the last 450 000 years of deap-sea core V12-122 (Carribean Sea). This picture was taken from Imbrie, Van Donk and Kipp (1974). The value of $\delta 18$ is proportional to the total ice volume on earth.

variations (as for example modelled by simple energy-balance climate models). The frequency distribution of the ice-sheet volume, however, shows a bimodal structure in accordance with our discussion. The observational data thus lend strong support to the validity of our results.

Summarizing, we conclude that the present way of including the feedback between ice-sheet height and mass balance reveals a very essential element of the climate system, namely, a cryosphere with two preferred modes.

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